Taylor Table To Derive a Generalized 4^{th} to 6^{th} Order Compact Pade Scheme

T. H. Pulliam

Stanford University

Taylor Tables

1. The generalized form of the equation is given by

$$\left(\frac{\partial u}{\partial x}\right)_{j-1} + \alpha\left(\frac{\partial u}{\partial x}\right)_{j} + \left(\frac{\partial u}{\partial x}\right)_{j+1} - \frac{A}{2\Delta x}\left(-u_{j-1} + u_{j+1}\right) - \frac{B}{4\Delta x}\left(-u_{j-2} + u_{j+2}\right) = er_t$$

1. The equation is written on terms of the single free coefficients α, A, B which must be determined using the Taylor Table approach as outlined below.
2. One goal is to find the value of α, A, B wich results in a 6^{th} Order Scheme
3. We can also defind a class of 4^{th} schemes where α is a free parameter and A, B are functions of α

Taylor Table

	u_j	$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3 \cdot \left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4 \cdot \left(\frac{\partial^4 u}{\partial x^4}\right)_j$	$\Delta x^5 \cdot \left(\frac{\partial^5 u}{\partial x^5}\right)_j$	$\Delta x^6 \cdot \left(\frac{\partial^6 u}{\partial x^6}\right)_j$	$\Delta x^7 \cdot \left(\frac{\partial^7 u}{\partial x^7}\right)$
$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_{j-1}$		1	$(-1)\,\frac{1}{1!}$	$(-1)^2\,\frac{1}{2!}$	$(-1)^3\,\frac{1}{3!}$	$(-1)^4\,\frac{1}{4!}$	$(-1)^5\,\frac{1}{5!}$	$(-1)^6\,\frac{1}{6!}$
$\Delta x \cdot \alpha \left(\frac{\partial u}{\partial x}\right)_j$		α						
$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_{j+1}$		1	$(1)\,\frac{1}{1!}$	$(1)^2\,\frac{1}{2!}$	$(1)^3\,\frac{1}{3!}$	$(1)^4\,\frac{1}{4!}$	$(1)^5\,\frac{1}{5!}$	$(1)^6\,\frac{1}{6!}$
$\frac{A}{2}\,u_{j-1}$	$\frac{A}{2}$	$\frac{A}{2}\,(-1)\,\frac{1}{1!}$	$\frac{A}{2}\,(-1)^2\,\frac{1}{2!}$	$\frac{A}{2}\,(-1)^3\,\frac{1}{3!}$	$\frac{A}{2}\,(-1)^4\,\frac{1}{4!}$	$\frac{A}{2}\,(-1)^5\,\frac{1}{5!}$	$\frac{A}{2}\,(-1)^6\,\frac{1}{6!}$	$\frac{A}{2}\,(-1)^7$
$-\frac{A}{2}\,u_{j+1}$	$-\frac{A}{2}$	$-\frac{A}{2}\,(1)\,\frac{1}{1!}$	$-\frac{A}{2}\,(1)^2\,\frac{1}{2!}$	$-\frac{A}{2}\,(1)^3\,\frac{1}{3!}$	$-\frac{A}{2}\,(1)^4\,\frac{1}{4!}$	$-\frac{A}{2}\,(1)^5\,\frac{1}{5!}$	$-\frac{A}{2}\,(1)^6\,\frac{1}{6!}$	$-\frac{A}{2}\,(1)^7$
$\frac{B}{4}\,u_{j-2}$	$\frac{B}{4}$	$\frac{B}{4}\,(-2)\,\frac{1}{1!}$	$\frac{B}{4}\,(-2)^2\,\frac{1}{2!}$	$\frac{B}{4}\,(-2)^3\,\frac{1}{3!}$	$\frac{B}{4}\,(-2)^4\,\frac{1}{4!}$	$\frac{B}{4}\,(-2)^5\,\frac{1}{5!}$	$\frac{B}{4}\,(-2)^6\,\frac{1}{6!}$	$\frac{B}{4}\,(-2)^7$
$-\frac{B}{4}\,u_{j+2}$	$-\frac{B}{4}$	$-\frac{B}{4}\,(2)\,\frac{1}{1!}$	$-\frac{B}{4}\,(2)^2\,\frac{1}{2!}$	$-\frac{B}{4}\,(2)^3\,\frac{1}{3!}$	$-\frac{B}{4}\,(2)^4\,\frac{1}{4!}$	$-\frac{B}{4}\,(2)^5\,\frac{1}{5!}$	$-\frac{B}{4}\,(2)^6\,\frac{1}{6!}$	$-\frac{B}{4}\,(2)^7$
$=$								
$\Delta x \cdot er_t$	0	0	0	0	0	?	0	?

Coefficient Equations

1. For Consistency and at least 4^{th} Order Accuracy, the first five columns are set to zero.

2. Note because of the skew symmetry of the original equation the odd numbers columms sum exact to 0

3. For 6^{th} Order Accuracy we need

$$\alpha + 2 - A - B = 0, \quad 1 - \frac{A}{6} - \frac{2B}{3} = 0, \quad 2 - \frac{A}{5} - \frac{16B}{5} = 0 \tag{1}$$

1. Solving we have $\alpha = 3$, $A = \frac{14}{3}$, and $B = \frac{1}{3}$

2. Then $er_t = -\frac{1}{180}\Delta x^6 \left(\frac{\partial^7 u}{\partial x^7}\right)_j$

Generalized Scheme

1. Instead of requiring a 6^{th} order scheme relax the conditions to allow the sixth column to be non-zero and find A, B as a function of α
2. Solving the first two relations for A, B we have $A = \frac{4+2\alpha}{3}$ and $B = \frac{4-\alpha}{3}$
3. For $\alpha = 3$:the above 5 point 6^{th} Order Scheme
4. For $\alpha = 4$: the 3 point 4^{th} Order Scheme
5. For $\alpha \neq 3$: a class of 4^{th} Order Schemes different characteristics.
6. In general,
$$er_t = \Delta x^4 \frac{1}{10} \left(1 - \frac{\alpha}{3}\right) \left(\frac{\partial^5 u}{\partial x^5}\right)_j + \Delta x^6 \frac{1}{1260} (8 - 5\alpha) \left(\frac{\partial^7 u}{\partial x^7}\right)_j$$
7. See Note on Modified Wave Number for General Pade Schemes